

AD-A080 000 FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS F/S 12/1
THE EXACT AND ASYMPTOTIC FORMULAS FOR THE STATE PROBABILITIES I--ETC(U)
DEC 79 M LACATO, M A LANGBERG DAAG29-79-C-0158
UNCLASSIFIED PSU-STATISTICS-H-830 ARO-16713.6-M NL

1 of 1
AD-A080 000



END
DATE
10 MAR 80
3-80

204

(18) HRB/

(17) 16713.6-N/

(12) LEVEL II

J. T. ...

6
THE EXACT AND ASYMPTOTIC FORMULAS
FOR THE STATE PROBABILITIES IN SIMPLE
EPIDEMICS WITH m KINDS OF SUSCEPTIBLES,
by

(19) H. Lacayo¹ and Naftali A. Langberg²

(14) FSU-Statistics Report-M-530, TR-142-ARM
USARO Technical Report No. D-43

(11) December 1979

(13) 17/

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

¹Research sponsored by the National Institute of Environmental Health Sciences, under Grant 5-T32-ES007011.

(15)
²Research supported by the United States Army Research Office, Durham, under Grant No. DAAG29-79-C0158, PHS-5-T32-ES007011

DDC
RECEIVED
FEB 15 1980
REGULATED
B

THE VIEW, OPINIONS, AND/OR FINDINGS CONTAINED IN THIS REPORT ARE THOSE OF THE AUTHOR(S) AND SHOULD NOT BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION, POLICY, OR DECISION, UNLESS SO DESIGNATED BY OTHER DOCUMENTATION.

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

y/B

498 277

The Exact And Asymptotic Formulas
For The State Probabilities In Simple
Epidemics With m Kinds Of Susceptibles

by

H. Lacayo and Naftali A. Langberg

ABSTRACT.

A population of susceptible individuals partitioned into m groups and exposed to a contagious disease is considered. It is assumed that an individual's susceptibility at time t depends on the number of susceptible individuals at time t in his group, and on the total number of infective individuals at time t .

The progress of this simple epidemic is modeled by an m -dimensional stochastic process. The components of this stochastic process represent the number of infective individuals in the respective groups at time t . Exact and approximate formulas for the joint and marginal state probabilities are obtained. It is shown that the approximate formulas are simple functions of time while, the derivations of the exact formulas involve tedious computations.

Key words: simple epidemics, different levels of susceptibility, convergence in distribution, exponential, negative binomial, and multinomial distributions.

ACCESSION for		
NTIS	White Section	<input checked="" type="checkbox"/>
DDC	Buff Section	<input type="checkbox"/>
UNANNOUNCED		<input type="checkbox"/>
JUSTIFICATION _____		
BY _____		
DISTRIBUTION/AVAILABILITY CODES		
Dist.	AVAIL. and/or	SPECIAL
A		

1. Introduction and Summary.

We say that a population of susceptible individuals (susceptibles), exposed to a contagious disease, (disease) undergoes a simple epidemic if the following four assumptions hold [Bailey (1975)].

- (1.1) At each point in time at most one susceptible contracts the disease.
- (1.2) Once a susceptible contracts the disease he remains contagious for the duration of the epidemic.
- (1.3) Individuals neither join nor do they depart from the population. And
- (1.4) All interactions between a susceptible and an infective individual (infective) are "equally likely" to result in an infection.

Gart (1968) models the progress of a simple yaws epidemic based on a data set obtained in New Guinea. He notices that the susceptibility level of an individual depends on his previous history of yaws. Gart divides the susceptibles into two groups: those with a positive yaws history, and those with a negative yaws history. He replaces, at least implicitly, Assumption (1.4) by:

- (1.5) All interactions between a susceptible that has a specified yaws history and an infective are "equally likely" to result in an infection.

In this paper we follow Gart (1972) and consider a population of susceptibles partitioned to m subpopulations, $m = 2, 3, \dots$, and exposed to a disease. We assume that the susceptibility level of an individual varies according to his membership in the various subpopulations.

Let T_0 be the first time there is at least one infective in each of the subpopulations, n_r be the number of susceptibles in the r^{th} subpopulation

at T_0 , b_r be the number of infectives in subpopulation r at T_0 , $r = 1, \dots, m$, and let $n = \sum_{r=1}^m n_r$. We describe the progress of the epidemic among susceptibles by an m -dimensional stochastic process: $\underline{X}_n(t) = [X_{n,1}(t), \dots, X_{n,m}(t)]$, $t \in [0, \infty)$. The components of $\underline{X}_n(t)$ represent the number of infectives in the respective subpopulations at time t measured from T_0 . Although the infection rates of the various subpopulations are not used explicitly we present them for reference purposes.

Definition 1.1. Let $r = 1, \dots, m$, and $t \in [0, \infty)$. Then the infection rate of subpopulation r at time t is given by

$$(1.6) \quad \lim_{h \rightarrow 0^+} h^{-1} P\{X_{n,r}(t+h) - X_{n,r}(t) = 1 | \underline{X}_n(t)\}.$$

We say that a population undergoes a simple Gart epidemic if Assumptions (1.1) through (1.3) and the following one are satisfied.

(1.7) The infection rate of subpopulation r at time t depends on the total number of infectives at time t , and on the number of susceptibles in subpopulation r at time t , $r = 1, \dots, m$, $t \in [0, \infty)$.

Note that Assumption (1.7) is the rigorous formulation of the statement: all interactions between a susceptible that belongs to a specified subpopulation and an infective are "equally likely" to result in an infection.

In Section 2 we construct a variety of m -dimensional stochastic processes. These processes can be used to model the progress of simple Gart epidemics among susceptibles. Section 3 contains formulas for the joint and marginal state probabilities: $P\{X_{n,r}(t) - b_r = k_r, r=1, \dots, m\}$, and $P\{X_{n,r}(t) - b_r = q\}$, $r = 1, \dots, m$, $t \in [0, \infty)$, where $k_1, \dots, k_m, q \in \{0, 1, \dots\}$. These formulas are calculated without the traditional use of the differential equations associated

with the state probabilities. This is done by utilizing a formula for the distribution function of a sum of independent exponential random variables (rva's) given by Billard, Lacayo, and Langberg (1979).

Let $\alpha_1, \dots, \alpha_m \in (0, \infty)$, and $X_n(t) = \sum_{r=1}^m X_{n,r}(t)$. Gart (1968), following the classical approach, assumes that the infection rates at time t of the subpopulations are given by

$$(1.8) \quad \alpha_r(n_r + b_r - X_{n,r}(t))X_n(t), \quad r = 1, \dots, m, \quad t \in [0, \infty).$$

These rates have the property, shown in Section 2, that the duration time of the simple Gart epidemic tends to zero as $n_\ell \rightarrow \infty$ for $\ell = 1, \dots, m$. To obtain approximations to the joint and marginal state probabilities when n_1, \dots, n_m are sufficiently large we adjust the above rates, and assume that the infection rates at time t of the subpopulations are given respectively by

$$(1.9) \quad n_r^{-1} \alpha_r(n_r + b_r - X_{n,r}(t))X_n(t), \quad r = 1, \dots, m, \quad t \in [0, \infty).$$

In Section 4, we obtain, under Assumption (1.9), approximations to the quantities: $P\{X_{n,r}(t) - b_r = k_r, r=1, \dots, m\}$, $P\{X_{n,r}(t) - b_r = q\}$, $P\{X_n(t) - \sum_{\ell=1}^m b_\ell = q\}$,

$EX_n(t)$, $EX_{n,r}(t)$, $\text{Var}\{X_n(t)\}$, and $\text{Var}\{X_{n,r}(t)\}$, when n_1, \dots, n_m are sufficiently large. These approximations are expressed as simple functions of t while, the derivations of the exact values of these quantities involve complicated computations, as we illustrate in Section 3.

2. Model Construction.

In this section we construct a variety of m -dimensional stochastic processes. These processes can be used to describe the progress of simple Galt epidemics among susceptibles.

First, we introduce some notation. Let $T_{n,k}$, the k^{th} interinfection time, be defined as the time that elapses between the $b+k-1$ and the $b+k$ infection, $k=1, \dots, n$. Let $S_{n,0} = 0$, $S_{n,k} = \sum_{q=1}^k T_{n,q}$, $k=1, \dots, n$, and $S_{n,n+1} = \infty$. Further, let $\xi_{n,0} = 0$, and $\xi_{n,k}$, $k=1, \dots, n$, be rva's assuming values in the set $\{1, \dots, m\}$. The rva $\xi_{n,k}$ specifies the subpopulation membership of the $b+k$ infective, $k=1, \dots, n$. Finally, let I be the indicator function, and $C_{n,k,r} = n_r - \sum_{q=0}^{k-1} I(\xi_{n,q}=r)$, $k=1, \dots, n$, $r=1, \dots, m$. Note that $C_{n,k,r} \geq C_{n,n,r} \geq 0$, $k=1, \dots, n$, $r=1, \dots, m$.

For $r=1, \dots, m$, $k=0, \dots, n$, and $t \in (0, \infty)$, the following event equality holds.

$$(2.1) \quad (X_{n,r}(t) - b_r = k) = \bigcup_{q=k}^n (S_{n,q} \leq t < S_{n,q+1}, \sum_{j=1}^q I(\xi_{n,j}=r) = k).$$

Thus, to construct the stochastic process $X_n(t)$ it suffices to determine the distribution function of the random vector (rve) $[T_{n,k}, \xi_{n,k}, k=1, \dots, n]$.

Next, we determine the distribution function of this rve. Let $a(n, k, j, r)$, $k=1, \dots, n$, $j=1, \dots, n$, $r=1, \dots, m$, be positive real numbers. Assume:

$$(2.2) \quad P\{\xi_{n,k}=r | \xi_{n,q}, q=0, \dots, k-1\} = \\ = a(n, k, C_{n,k,r}, r) \left\{ \sum_{\ell=1}^m a(n, k, C_{n,k,\ell}, \ell) \right\}^{-1}, \quad k=0, \dots, n, \\ r=1, \dots, m, \text{ and};$$

(2.3) The conditional rva's $\{T_{n,k} | \xi_{n,q}, q = 0, \dots, k-1\}$, $k = 1, \dots, n$, are independent exponentially distributed with means equal respectively to $\{\sum_{\ell=1}^m a(n,k, C_{n,k,\ell}, \ell)\}^{-1}$.

Clearly, Assumptions (2.2) and (2.3) determine the distribution function of the rve $[T_{n,k}, \xi_{n,k}, k=1, \dots, n]$.

Note that by the memoryless property of exponential rva's [Barlow, Proschan (1975), p. 56], Equation (2.1), and Assumptions (2.2), (2.3), the infection rates at time t of the processes constructed in this section are equal to $a(n, X_n(t) - b + 1, n_r + b_r - X_{n,r}(t), r)$, $r = 1, \dots, m$, $t \in (0, \infty)$. Thus, these processes can be used to describe the progress of simple Gart epidemics among susceptibles.

The classical model, considered by Gart (1968) and (1972), follows by setting $a(n,k,j,r) = \alpha_r (b+k-1)j$. By Assumption (2.2) the duration time of the classical simple Gart epidemic is equal to $E\{\sum_{q=1}^n (b+q-1) \sum_{\ell=1}^m \alpha_r C_{n,q,r}\}^{-1}$. Thus, it is less than or equal to $(\min_{1 \leq \ell \leq m} \alpha_\ell)^{-1} \sum_{q=1}^n [(b+q-1)(n-q+1)]^{-1}$. Consequently, the duration time of classical simple Gart epidemic tends to zero as $n_\ell \rightarrow \infty$ for $\ell = 1, \dots, m$. In Section 4 we investigate the asymptotic behaviour of simple Gart epidemic models determined by letting $a(n,k,j,r) = n_r^{-1} \alpha_r (b+k-1)j$. Following Severo (1969) we can define $a(n,k,j,r) = \alpha_r n_r^\delta (b+k-1)^\theta j^\lambda$, $\delta \in (-\infty, \infty)$, $\theta, \lambda \in [0, \infty)$, and thus, construct extensions to the simple epidemic models used and motivated by him. In particular for $\delta = 0$, $\theta = \lambda = 1/2$, we get infection rates used by McNeil (1972) to describe simple epidemic situations.

3. Formulas for the State Probabilities.

Let $q, k_1, \dots, k_m \in \{0, \dots, n\}$, $\underline{k} = [k_1, \dots, k_m]$, and $k = \sum_{r=1}^m k_r$.

Throughout we assume that $0 \leq k \leq n$. Further, let U_1, U_2, \dots , be a sequence of independent exponential rva's with means equal to 1, and μ_1, μ_2, \dots , be a sequence of positive real numbers.

This section contains formulas for the joint and marginal state probabilities: $P_{n,\underline{k}}(t) = P\{X_{-n}(t) - b = \underline{k}\}$, and $P_{n,q,r}(t) = P\{X_{n,r}(t) - b_r = q\}$, $r = 1, \dots, m, t \in (0, \infty)$. These formulas are calculated without the traditional use of the differential equations associated with the state probabilities. Rather, we utilize an available formula for the distribution function of a sum of independent exponential rva's. For the sake of completeness we present this formula.

Theorem 3.1. [Billard, Lacayo and Langberg (1979), Theorem 1]. Let M be a positive integer, and $\ell_M(j) = (-1)^{M-1} \sum_{j_1 + \dots + j_M = j} \prod_{q=1}^M (\mu_q)^{j_q+1}$,

$j = M, M+1, \dots$. Then for $t \in (0, \infty)$.

$$(3.1) \quad P\left\{\sum_{q=1}^M \mu_q^{-1} U_q \leq t\right\} = \sum_{j=M}^{\infty} \ell_M(j) (-t)^j / (j!).$$

To aid in computing the joint and marginal state probabilities we introduce the following notation. Let $\underline{\ell}_k = [\ell_0, \dots, \ell_k]$, $\ell_0 = 0$, $\ell_1, \dots, \ell_k \in \{1, \dots, m\}$, $B_{\underline{k}} = \{\underline{\ell}_k : \sum_{q=0}^k I(\ell_q = r) = k_r, r=1, \dots, m\}$, and $A_{r,j,q} = \{\underline{\ell}_j : \sum_{e=0}^j I(\ell_e = r) = q\}$, $j=q, q+1, \dots$. By Equation (2.1) we obtain that for $t \in (0, \infty)$

$$(3.2) \quad P_{n,\underline{k}}(t) = \sum_{\underline{\ell}_k} P\{S_{n,k} \leq t < S_{n,k+1} | \xi_{n,q} = \ell_q, q=0, \dots, k\} P\{\xi_{n,q} = \ell_q, q=0, \dots, k\} I(\underline{\ell}_k \in B_{\underline{k}}),$$

and that

$$(3.3) \quad P_{n,q,r}(t) =$$

$$= \sum_{j=q}^n \sum_{\underline{\ell}_j} P\{S_{n,j} \leq t < S_{n,j+1} \mid \xi_{n,e} = \ell_e, e=0, \dots, j\} P\{\xi_{n,e} = \ell_e, e=0, \dots, j\} I(\underline{\ell}_j \in A_{r,j,q}).$$

Thus, to compute $P_{n,k}(t)$ and $P_{n,q,r}(t)$ it suffices to evaluate $P\{\xi_{n,q} = \ell_q, q=0, \dots, k\}$ and $P\{S_{n,k} \leq t < S_{n,k+1} \mid \xi_{n,q} = \ell_q, q=0, \dots, k\}$.

Now, we present formulas for these probabilities. Let

$$D_{n,k,r}(\underline{\ell}_k, q) = n_r - \sum_{j=0}^{q-1} I(\ell_j = r), \quad q = 1, \dots, k+1, \text{ and}$$

$$n(n, k, \underline{\ell}_k, q) = \sum_{r=1}^m a(n, q, D_{n,k,r}(\underline{\ell}_k, q), r), \quad q = 1, \dots, k+1.$$

First, by Assumption (2.2)

$$(3.4) \quad P\{\xi_{n,q} = \ell_q, q=0, \dots, k\} =$$

$$= \prod_{q=1}^k [\{n(n, k, \underline{\ell}_k, q)\}^{-1} \prod_{r=1}^m \{a(n, q, D_{n,k,r}(\underline{\ell}_k, q), r)\}^{I(\ell_q = r)}].$$

Next, let $f_k(\mu_1, \dots, \mu_k, t)$ be the density function of the rva $\sum_{q=1}^k \mu_q^{-1} U_q$. Then for $k = 1, 2, \dots$, and $t \in (0, \infty)$

$$(3.5) \quad P\{\sum_{q=1}^k \mu_q^{-1} U_q \leq t < \sum_{q=1}^{k+1} \mu_q^{-1} U_q\} =$$

$$= \int_0^t e^{-\mu_{k+1}(t-u)} f_k(\mu_1, \dots, \mu_k, u) du = \mu_{k+1}^{-1} f_{k+1}(\mu_1, \dots, \mu_{k+1}, t).$$

Thus, by Assumption (2.3) and Theorem 3.1

$$\begin{aligned}
 (3.6) \quad & P\{S_{n,k} \leq t < S_{n,k+1} | \xi_{n,q} = \ell_q, q=0, \dots, k\} = \\
 & = \begin{cases} e^{-\eta(n,k,\underline{\ell}_k,1)t} & k = 0 \\
 \sum_{j=n}^{\infty} \frac{(-1)^{n+j-1} t^j}{j!} \sum_{j_1+\dots+j_n=j} \prod_{q=1}^n \{\eta(n,k,\underline{\ell}_k,q)\}^{j_q+1} & k = n \\
 \{\eta(n,k,\underline{\ell}_k,k+1)\}^{-1} \sum_{j=k+1}^{\infty} \frac{(-1)^{k+j} t^{j-1}}{(j-1)!} \sum_{j_1+\dots+j_{k+1}=j} \prod_{q=1}^{k+1} \{\eta(n,k,\underline{\ell}_k,q)\}^{j_q+1} & 1 \leq k < n. \end{cases}
 \end{aligned}$$

Consequently, the formulas for the joint and marginal state probabilities can be obtained from (3.2) and (3.3) by substitution.

4. The Asymptotic Behaviour.

This section contains our main asymptotic results. All limits are calculated as $n_\ell \rightarrow \infty$ for $\ell = 1, \dots, m$.

For the sake of completeness we present two definitions.

Definition 4.1. Let $\ell \in \{1, 2, \dots\}$, and $p \in (0, 1)$. We say that the rva Y has a negative binomial distribution with parameters ℓ , and p , and write $Y \sim \text{NB}(\ell, p)$ if

$$(4.1) \quad P\{Y=q\} = \binom{\ell+q-1}{q} p^\ell (1-p)^q, \quad q = 0, 1, \dots$$

Definition 4.2. Let $e, \ell \in \{1, 2, \dots\}$, and $\gamma_1, \dots, \gamma_e \in (0, 1)$ such that $\sum_{j=1}^e \gamma_j = 1$. We say that the rve $\underline{W}_e = \{W_1, \dots, W_e\}$ has a multinomial distribution with parameters $\ell, \gamma_1, \dots, \gamma_e$, and write $\underline{W}_e \sim \text{MN}(\ell, \gamma_1, \dots, \gamma_e)$ if

$$(4.2) \quad P\{W_j = \ell_j, j=1, \dots, e\} = (\ell! \prod_{j=1}^e \gamma_j^{\ell_j}) / (\prod_{j=1}^e \ell_j!),$$

$$\ell_1, \dots, \ell_e \in \{0, \dots, \ell\}, \quad \sum_{j=1}^e \ell_j = \ell.$$

Let $\alpha = \sum_{r=1}^m \alpha_r$, $\beta_r(t) = e^{-\alpha t} [1 - (1 - e^{-\alpha t})(1 - \alpha_r \alpha^{-1})]^{-1}$, $Y(t) \sim \text{NB}(b, e^{-\alpha t})$, $Y_r(t) \sim \text{NB}(b, \beta_r(t))$, and $\underline{W}_m(k) \sim \text{MN}(k, \alpha_1 \alpha^{-1}, \dots, \alpha_m \alpha^{-1})$, $r = 1, \dots, m$, $k = 1, 2, \dots, t \in (0, \infty)$.

Assume throughout that $a(n, k, j, r) = n_r^{-1} \alpha_r (b+k-1)j$, $j = 1, \dots, n_r$, $r = 1, \dots, m$, $k = 1, \dots, n$, $n = 1, 2, \dots$. First, we show that for $t \in (0, \infty)$

$$(4.3) \quad \lim_{n, \underline{k}} P_{n, \underline{k}}(t) = P\{Y(t) = \sum_{r=1}^m k_r\} P\{\underline{W}_m(\sum_{r=1}^m k_r) = \underline{k}\},$$

$$(4.4) \quad \lim P\{X_n(t) - b = q\} = P\{Y(t) = q\}, \text{ and}$$

$$(4.5) \quad \lim_{n, q, r} P_{n, q, r}(t) = P\{Y_r(t) = q\}, \quad r = 1, \dots, m.$$

Next, we show that for $t, \beta \in (0, \infty)$

$$(4.6) \quad \lim E\{X_n(t)\}^\beta = E\{Y(t)\}^\beta, \text{ and that}$$

$$(4.7) \quad \lim E\{X_{n,r}(t)\}^\beta = E\{Y_r(t)\}^\beta, \quad r = 1, \dots, m.$$

In particular it follows from Statements (4.6) and (4.7) that for $t \in (0, \infty)$

$$(4.8) \quad \lim EX_n(t) = b(e^{\alpha t} - 1),$$

$$(4.9) \quad \lim \text{Var}\{X_n(t)\} = b(e^{2\alpha t} - e^{\alpha t}),$$

$$(4.10) \quad \lim EX_{n,r}(t) = b((\beta_r(t))^{-1} - 1), \quad r = 1, \dots, m, \text{ and}$$

$$(4.11) \quad \lim \text{Var}\{X_{n,r}(t)\} = b((\beta_r(t))^{-2} - (\beta_r(t))^{-1}), \quad r = 1, \dots, m.$$

Note that the approximate formulas given in Statements (4.3) through (4.5), and in Statement (4.8) through (4.11) are indeed simple functions of time.

The following three lemmas are needed to prove Statements (4.3) through (4.5).

Lemma 4.1. Let $k \in \{1, 2, \dots\}$, and $\ell_1, \dots, \ell_k \in \{1, \dots, m\}$. Then

$$(4.12) \quad \lim P\{\xi_{n,q} = \ell_q, q=1, \dots, k\} = \prod_{r=1}^m (\alpha_r \alpha^{-1})^{\sum_{q=1}^k I(\ell_q=r)}.$$

Proof. Let $\ell_0 = 0$. Then $P\{\xi_{n,q} = \ell_q, q=1, \dots, k\} = \prod_{q=1}^k P\{\xi_{n,q} = \ell_q | \xi_{n,j} = \ell_j, j=0, \dots, q-1\}$. Consequently the result of the lemma follows by Assumption (2.2). ||

Lemma 4.2. Let $k \in \{1, 2, \dots\}$, $\ell_1, \dots, \ell_k \in \{1, \dots, m\}$, and $\ell_0 = 0$. Then the sequence of conditional rva's: $\{S_{n,k} | \xi_{n,q} = \ell_q, q=0, \dots, k-1\}$, $n = 1, 2, \dots$, converges in distribution to the rva $\sum_{q=1}^k \alpha^{-1} (b+q-1)^{-1} U_q$

Proof. To prove the result of the lemma it suffices, by the Cramer-Wold device [Billingsley (1968), p. 49], to show that the sequence of conditional rve's: $\{T_{n,1}, \dots, T_{n,k} | \xi_{n,q} = \ell_q, q=0, \dots, k-1\}$, $n = 1, 2, \dots$, converges in distribution to the rve $\{\alpha^{-1}b^{-1}U_1, \dots, \alpha^{-1}(b+k-1)^{-1}U_k\}$.

The preceding statement follows from Assumption (2.3). ||

Lemma 4.3. Let d, k , be two positive integers. Further, let $\{U_{1:d+k-1}, \dots, U_{d+k-1:d+k-1}\}$ be the order statistic of a sample of size $d + k - 1$ taken from the population U_1 . Then the rva's $\sum_{q=1}^k (d+q-1)^{-1}U_q$ and $U_{k:d+k-1}$ are equal in distribution.

Proof. Let $U_{0,d+k-1} = 0$. Then the spacings: $U_{k-q+1:d+k-1} - U_{k-q:d+k-1}$, $q = 1, \dots, k$, are independent exponentially distributed rva's with means respectively equal to $(d+q-1)^{-1}$ [Barlow Proschan (1975) p. 59]. Thus, the rva's $\sum_{q=1}^k (d+q-1)^{-1}U_q$ and $\sum_{q=1}^k \{U_{k-q+1:d+k-1} - U_{k-q:d+k-1}\}$ are equal in distribution. To complete the proof of the lemma we note that

$$\sum_{q=1}^k \{U_{k-q+1:d+k-1} - U_{k-q:d+k-1}\} = U_{k:d+k-1}. \quad ||$$

We are ready now to prove Statements (4.3) through (4.5). First, we prove Statement (4.3).

Theorem 4.4. Let $k_1, \dots, k_m \in \{0, 1, \dots\}$, $\underline{k} = \{k_1, \dots, k_m\}$, $k = \sum_{r=1}^m k_r$, and $t \in (0, \infty)$. Then $\lim_{n, \underline{k}} P_{n, \underline{k}}(t) = P\{Y(t) = k\}P\{\underline{W}_m(k) = \underline{k}\}$.

Proof. Let B_k be defined as in Section 3, and let $\underline{\ell}_k \in B_k$. Then by Lemma 4.1 $\lim P\{\xi_{n,q} = \ell_q, q=1, \dots, k\} = \prod_{r=1}^m (\alpha_r \alpha^{-1})^{k_r}$. Further, by Lemmas 4.2, 4.3, and equation (3.5) $\lim P\{S_{n,k} \leq t < S_{n,k+1} | \xi_{n,q} = \ell_q, q = 0, \dots, k\} =$

$$P\{\sum_{q=1}^k (b+q-1)^{-1}U_q \leq t < \sum_{q=1}^{k+1} (b+q-1)^{-1}U_q\} = (b+k)^{-1} \binom{b+k}{k+1} (k+1)e^{-abt} (1-e^{-at})^k.$$

Finally, we note that B_k contains $k! / (\prod_{r=1}^m k_r!)$ elements. Consequently, the

result of the theorem follows from Equation (3.2). ||

Let $t \in (0, \infty)$, and $\underline{Z}_m(t) = \{Z_1(t), \dots, Z_m(t)\}$ be a rve such that $\{Z_m | \sum_{r=1}^m Z_r(t)\} \sim MN(\sum_{r=1}^m Z_r(t), \alpha_1 \alpha^{-1}, \dots, \alpha_m \alpha^{-1})$, and $\sum_{r=1}^m Z_r(t) \sim NB(b, e^{-\alpha t})$.

By Theorem 4.4 and a well known result [Billingsley (1968), p. 16] we conclude,

Corollary 4.5. Let $t \in (0, \infty)$. Then the sequence of rve's $\underline{X}_n(t) - \underline{b}$, $n = 1, 2, \dots$, converges in distribution to the rve $\underline{Z}_m(t)$.

From Corollary 4.5 and the Cramer-Wold device we obtain,

Corollary 4.6. Let $r = 1, \dots, m$, and $t \in (0, \infty)$. Then the two sequences of rva's $X_n(t) - b$, and $X_{n,r}(t) - b_r$, $n = 1, 2, \dots$, converge in distribution to $\sum_{\ell=1}^m Z_\ell(t)$ and $Z_r(t)$ respectively.

Now, we prove Statements (4.4) and (4.5).

Theorem 4.7. Let $q \in \{0, 1, \dots\}$, $r = 1, \dots, m$, and $t \in (0, \infty)$. Then

(a) $\lim P\{X_n(t) - b = q\} = P\{Y(t) = q\}$, and (b) $\lim P_{n,q,r}(t) = P\{Y_r(t) = q\}$.

Proof. Part (a) follows clearly from Corollary 4.6. To prove part (b) it suffices by Corollary 4.6 to evaluate $P\{Z_r(t) = q\}$.

$$\begin{aligned} & \text{It is well known that for } k = 1, 2, \dots, P\{Z_r(t) = q | \sum_{\ell=1}^m Z_\ell(t) = k\} = \\ & \binom{k}{q} (\alpha_r \alpha^{-1})^q (1 - \alpha_r \alpha^{-1})^{k-q}, \quad q = 0, \dots, k. \text{ Thus, } P\{Z_r(t) = q\} = \\ & = \sum_{k=q}^{\infty} \binom{b+k-1}{k} \binom{k}{q} e^{-\alpha b t} (1 - e^{-\alpha t})^k (\alpha_r \alpha^{-1})^q (1 - \alpha_r \alpha^{-1})^{k-q} = \\ & = [q!(b-1)!]^{-1} e^{-\alpha b t} [(1 - e^{-\alpha t}) \alpha_r \alpha^{-1}]^q \sum_{k=q}^{\infty} (b+k-1)! [(k-q)!]^{-1} [(1 - e^{-\alpha t}) (1 - \alpha_r \alpha^{-1})]^{k-q} \\ & = [q!(b-1)!]^{-1} e^{-\alpha b t} [(1 - e^{-\alpha t}) \alpha_r \alpha^{-1}]^q \int_0^{\infty} e^{-u} u^{b+q-1} \sum_{k=q}^{\infty} [(k-q)!]^{-1} [u(1 - e^{-\alpha t}) (1 - \alpha_r \alpha^{-1})]^{k-q} du \\ & = \binom{b+q-1}{q} (\beta_r(t))^b (1 - \beta_r(t))^q. \quad || \end{aligned}$$

Now we determine the convergence of the moments.

Theorem 4.8. Let $t, \beta \in (0, \infty)$. Then $\lim E\{X_n(t)\}^\beta = E\{Y(t)\}^\beta$.

Proof. First, we note that

$$\begin{aligned}
 (4.13) \quad E\{X_n(t)\}^\beta &= \beta \int_0^\infty y^{\beta-1} P\{X_n(t) > y\} dy = \\
 &= \beta \int_0^b y^{\beta-1} dy + \beta \int_0^\infty (b+z)^{\beta-1} P\{X_n(t) - b > z\} dz = \\
 &= b^\beta + \sum_{q=1}^\infty [(b+q)^\beta - (b+q-1)^\beta] P\{X_n(t) - b \geq q\}.
 \end{aligned}$$

Next, for $k = 1, 2, \dots$, $\sum_{r=1}^m n_r^{-1} \alpha_{r,n,k,r} \leq \alpha$, and for $q = 1, 2, \dots$, $b + q - 1 \leq 2bq$. Thus, by Lemma 4.3

$$\begin{aligned}
 (4.14) \quad P\{X_n(t) - b \geq k\} &= P\{S_{n,k} \leq t\} \leq P\left\{\sum_{q=1}^k q^{-1} U_q \leq 2abt\right\} = \\
 &= (1 - e^{-2abt})^k, \quad k = 1, 2, \dots
 \end{aligned}$$

Finally, we note that

$$(4.15) \quad \sum_{q=1}^\infty [(b+q)^\beta - (b+q-1)^\beta] (1 - e^{-2abt})^q \leq (2b)^\beta \sum_{q=1}^\infty q^\beta (1 - e^{-2abt})^q < \infty.$$

Consequently, the result of the theorem follows by Corollary 4.6 and the dominated convergence theorem [Loève (1963), p. 125]. ||

Finally, we prove Statement (4.7).

Theorem 4.9. Let $r = 1, \dots, m$, and $t, \beta \in (0, \infty)$. Then

$$\lim E\{X_{n,r}(t)\}^\beta = E\{Y_r(t)\}^\beta.$$

Proof. Note that $E\{X_{n,r}(t)\}^\beta =$

$$\begin{aligned}
 &= b_r^\beta + \sum_{q=1}^\infty [(b_r+q)^\beta - (b_r+q-1)^\beta] P\{X_{n,r}(t) - b_r \geq q\}, \text{ that by Inequality (4.14)} \\
 &P\{X_{n,r}(t) - b_r \geq q\} \leq P\{X_n(t) - b \geq q\} \leq (1 - e^{-2abt})^q, \quad q = 1, 2, \dots, \text{ and that} \\
 &\sum_{q=1}^\infty [(b_r+q)^\beta - (b_r+q-1)^\beta] (1 - e^{-2abt})^q \leq (2b_r)^\beta \sum_{q=1}^\infty q^\beta (1 - e^{-2abt})^q < \infty.
 \end{aligned}$$

Consequently, the result of the theorem follows by Corollary 4.6 and the dominated convergence theorem. ||

References.

- (1) Bailey, N.T.J. (1975). The mathematical theory of infectious diseases. Hafner, New York.
- (2) Barlow, R. and Proschan, F. (1975). Statistical theory of reliability and life testing probability models. Holt, Rinehart and Winston.
- (3) Billard, L. Lacayo, H. and Langberg, N.A. (1979). Generalizations of the simple epidemic process. To appear J. App. Prob.
- (4) Billingsley, P. (1968). Convergence of probability measures. John Wiley and Sons, New York.
- (5) Gart, J.J. (1968). The mathematical analysis of an epidemic with two kinds of susceptibles. Biometrics, 24, 557-566.
- (6) Gart, J.J. (1972). The statistical analysis of chain-binomial epidemic models with several kinds of susceptibles. Biometrics 28, 921-930.
- (7) Loève, M. (1963). Probability theory. D. Van Nostrand Co.
- (8) McNeil, D.R. (1972). On the simple stochastic epidemic. Biometrika 59, 494-497.
- (9) Severo, N.C. (1969). Generalizations of some stochastic epidemic models. Math Biosci. 4 395-402.

UNCLASSIFIED

Security Classification of this Page

REPORT DOCUMENTATION PAGE

1. REPORT NUMBERS FSU No. M530	2. GOVT. ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE The Exact and Asymptotic Formulas for the State Probabilities in Simple Epidemics with m Kinds of Susceptibles		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER FSU Statistics Report M530
7. AUTHOR(s) H. Lacayo and Naftali A. Langberg		8. CONTRACT OR GRANT NUMBER(s) USARO DAAG29-79 C0158 NIEHS 5-T32-ES007011
9. PERFORMING ORGANIZATION NAME & ADDRESS The Florida State University Department of Statistics Tallahassee, Florida 32306		10. PROGRAM ELEMENT, PROJECT, TASK AREA AND WORK UNIT NOS.
11. CONTROLLING OFFICE NAME & ADDRESS United States Army Research Office, Durham P.O. Box 12211 Research Triangle Park, N.C. 27709		12. REPORT DATE December 1979
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release: distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS Simple epidemics, different levels of susceptibility, convergence in distribution, exponential, negative binomial, and multinomial distributions		
20. ABSTRACT A population of susceptible individuals partitioned into m groups and exposed to a contagious disease is considered. It is assumed that an individual's susceptibility at time t depends on the number of susceptible individuals at time t in his group, and on the total number of infective individuals at time t. The progress of this simple epidemic is modeled by an m-dimensional stochastic process. The components of this stochastic process represent the number of infective individuals in the respective groups at time t. Exact and approximate formulas for the joint and marginal state probabilities are obtained. It is shown that the approximate formulas are simple functions of time while, the derivations of the exact formulas involve tedious computations.		